

COMPUTABILITY OF FØLNER SETS

MATTEO CAVALERI

ABSTRACT. We define the notion of computability of Følner sets for finitely generated amenable groups. We prove, by an explicit description, that the Kharlampovich groups, finitely presented solvable groups with unsolvable Word Problem, have computable Følner sets. We also prove computability of Følner sets for extensions -with subrecursive distortion functions- of amenable groups with solvable Word Problem by finitely generated groups with computable Følner sets. Moreover we obtain some known and some new upper bounds for the Følner function for these particular extensions.

INTRODUCTION

In this paper we define and study an effective version of amenability for finitely generated groups in terms of computability of Følner sets. Let Γ be a group generated by a finite set X . We denote by $\pi_{\Gamma,X}: \mathbb{F}_X \rightarrow \Gamma$ the canonical epimorphism from the free group over X to Γ . For any $n \in \mathbb{N}$, an n -Følner set for Γ (with respect to X) is defined to be a non-empty finite subset $F \subset \Gamma$ such that

$$(1) \quad \frac{|F \setminus xF|}{|F|} \leq n^{-1}, \quad \forall x \in X.$$

Γ is *amenable* if it admits n -Følner sets, for all $n \in \mathbb{N}$. In order to make *effective* this notion, we simply ask for computability of finite preimages of Følner sets in the covering free group.

Definition A. Γ has *computable Følner sets* with respect to X if there exists an algorithm with:

INPUT: $n \in \mathbb{N}$

OUTPUT: $F \subset \mathbb{F}_X$ finite, such that $\pi_{\Gamma,X}(F)$ is n -Følner for Γ .

This definition does not depend on the particular choice of the finite set of generators (see Proposition 1).

The *Følner function* $F_{\Gamma,X}$ of Γ (with respect to X) was defined by Vershik [19] by

$$F_{\Gamma,X}(n) := \min\{|F| : F \subset \Gamma \text{ is } n\text{-Følner}\}.$$

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The aim of our investigation, and in particular of Definition A, is to formalize and answer an old question, in our notation:

Question (Vershik). “is it possible, in some sense, to (algorithmically) describe the n -Følner sets of Γ even if there is no solution for the Word Problem?”

This question arose after the construction by Kharlampovich of finitely presented groups, solvable and therefore amenable, with unsolvable Word Problem [11] (following the notation in [13], let denote them by $G(M)$).

Indeed a finitely generated amenable group with solvable Word Problem has computable Følner sets: for every $n \in \mathbb{N}$ we can enumerate all finite subsets of \mathbb{F}_X and for each subset check, by solvability of the Word Problem, condition (1), until we find the preimage of an n -Følner set: the algorithm will eventually stop because Γ is amenable (for details [2, 3]).

A first negative answer to Vershik’s question was given by Erschler [7], providing examples of finitely generated groups with Følner functions growing faster than any given function (a result recovered in [10, 17]): thus, when the given function is not subrecursive (i.e. without any recursive upper bound) there is no hope to algorithmically describe Følner sets of the associated groups (for details [2, 3]).

On the other hand, in [3], we proved that, if Γ is recursively presentable and amenable then

- (i) the Følner function of Γ is subrecursive;
- (ii) there exists an algorithm computing the Reiter functions of Γ ;
- (iii) computability of one-to-one preimages of Følner sets is equivalent to solvability of the Word Problem.

As a consequence, for recursively presented amenable groups, the notion of computability of Følner sets is the only non trivial notion of effective amenability that is not characterized by solvability of the Word Problem.

Indeed computability of Følner sets does not imply solvability of the Word Problem: this is the first consequence of the following theorem, proved by a very explicit description of the Følner sets of $G(M)$ (see Section 1).

Theorem A. *The Kharlampovich groups $G(M)$ have computable Følner sets.*

Amenability is stable under semidirect products and, more generally, under extensions: in the literature, the most common proofs of these two facts do not use the characterization of amenability by Følner sets. The book [5] is one of the exceptions and it was a valuable resource for our proofs. Moreover in [12] it was explicitly shown that a Følner net for the semidirect product

is given by the product of the Følner nets of the factor groups. However this does not yield an effective procedure to produce, for a fixed $n \in \mathbb{N}$, an n -Følner set.

After the Preliminaries, each section consists of a Theorem about the general shape of n -Følner sets of group extensions, a Corollary about computability of these Følner sets and a Corollary about the Følner functions. We can interpret the results of this paper as stability properties of the class of groups with computable Følner sets and of the class of groups with subrecursive Følner functions.

Section 1. We consider the case of a splitting extension by an Abelian group which is finitely generated as a normal subgroup (this is the case of $G(M)$): computability of Følner sets and subrecursivity of Følner function are preserved.

Section 2. We consider general Abelian extensions: subrecursivity of Følner functions is preserved but we prove computability of Følner sets just if the quotient group has solvable Word Problem. We don't know if this hypothesis is necessary. Asymptotically equivalent bounds for the Følner function of solvable groups could be also deduced from [7, 8], or using the comparison with the Følner function in free solvable groups in [18].

Section 3. We consider the semidirect product between two finitely generated groups: if both have computable Følner sets then the product has computable Følner sets.

Section 4. We consider an extension Γ of a finitely generated group K by a finitely generated group $N = \langle Y \rangle$. If K and N have subrecursive Følner functions and the *distortion function* $\Delta_N^\Gamma(n) := \max\{|\omega|_Y : \omega \in N, |\omega|_X \leq n\}$ is subrecursive, then Γ has subrecursive Følner function; if N has computable Følner sets, K is amenable with solvable word problem and Δ_N^Γ is subrecursive then Γ has computable Følner sets. Notice that it is possible that Δ_N^Γ is not subrecursive, see for example [1], even for solvable groups, see [6]. Again we don't know if these hypotheses are necessary.

Questions.

1. Have all finitely generated solvable groups computable Følner sets?
2. Is computability of Følner sets stable under quotients?
3. Does subrecursivity of the Følner function imply computability of Følner sets?

A positive answer to the third question would imply both computability of Følner sets for every recursively presented amenable group and a positive answer to the second question, because subrecursivity of the Følner function is stable under quotients (see [7, Lemma 2.2]); a positive answer

to the second question would imply a positive answer to the first one, because free solvable groups have solvable Word Problem and therefore have computable Følner sets.

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PRELIMINARIES

Throughout this paper $\mathfrak{Fol}_{\Gamma, X}(n)$ is the family of n -Følner sets of Γ with respect to X . For an element $g \in \Gamma$ we denote with $|g|$ the length with respect to $X \cup X^{-1}$ (so it is the minimal length of a word in \mathbb{F}_X representing g). For a different set of generators, say Y , we explicitly write $|g|_Y$. For a subset A we also denote with $|A|_Y$ the maximal length of the elements of A with respect to Y . We denote by B_n the ball of radius n in the free group and $B_n(\Gamma) := \pi_{\Gamma, X}(B_n)$ the ball of radius n of Γ .

Lemma 1. *For any $F \in \mathfrak{Fol}_{\Gamma, X}(n)$ and for all $g \in \Gamma$ we have:*

$$\frac{|F \setminus gF|}{|F|} \leq |g|n^{-1}.$$

Proof. At first we observe that if $F \in \mathfrak{Fol}_{\Gamma, X}(n)$, for every $x \in X$ we have:

$$\frac{|F \setminus x^{-1}F|}{|F|} = \frac{|x^{-1}(xF \setminus F)|}{|F|} = \frac{|xF \setminus F|}{|F|} = \frac{|F \setminus xF|}{|F|} \leq n^{-1}.$$

If $g = x_1 \dots x_{|g|}$, with $x_1, \dots, x_{|g|} \in X \cup X^{-1}$,

$$(F \setminus x_1 \dots x_{|g|}F) \subset [(F \setminus x_1F) \cup (x_1F \setminus x_1x_2F) \cup \dots \cup (x_1 \dots x_{|g|-1}F \setminus x_1 \dots x_{|g|}F)],$$

and $|x_1 \dots x_{j-1}F \setminus x_1 \dots x_jF| = |F \setminus x_jF|$. □

Proposition 1. *Suppose Γ has computable Følner sets with respect to X . Let $Y \subset \Gamma$ be another finite generating subset. Then Γ has computable Følner sets with respect to Y as well.*

Proof. By expressing every $x \in X$ in terms of words in Y , we define a homomorphism $\phi: \mathbb{F}_X \rightarrow \mathbb{F}_Y$ such that the following diagram commutes:

$$\begin{array}{ccc}
\mathbb{F}_X & \xrightarrow{\phi} & \mathbb{F}_Y \\
& \searrow \pi_{\Gamma,X} & \downarrow \pi_{\Gamma,Y} \\
& & \Gamma
\end{array}$$

We define the natural number $m := |\phi(X)|_Y$ as the maximum of the word length in \mathbb{F}_Y of the image of generators of \mathbb{F}_X . Suppose $W \subset \mathbb{F}_X$ is a finite subset such that $\pi_{\Gamma,X}(W) \in \mathfrak{Fol}_{\Gamma,X}(mn)$; by Lemma 1 $\pi_{\Gamma,Y}(\phi(W)) = \pi_{\Gamma,X}(W) \in \mathfrak{Fol}_{\Gamma,Y}(n)$. Combining the algorithm of Definition A for X , with the algorithm computing the homomorphism ϕ , we deduce the existence of the desired algorithm for Y . \square

Remark 1. In finitely presented case (by Tietze transformations), or, more generally, whenever we can write the old generators in terms of the new ones, we can explicitly update the algorithm of Definition A for the new generators.

Lemma 2. *If Γ is amenable then there exists $F \in \mathfrak{Fol}_{\Gamma,X}(n)$ such that $|F| \leq F_{\Gamma,X}(|X|n)$ and $F \subset B_{|F|}(\Gamma)$.*

Proof. We define

$$\mathfrak{Fol}'_{\Gamma,X}(n) := \{F, \text{ non-empty finite subset of } \Gamma : \frac{|\partial_X F|}{|F|} \leq \frac{1}{n}\},$$

where $\partial_X F := \{f \in F : \exists x \in X : xf \notin F\}$.

It is known and easy to see that if F' is of minimal cardinality in $\mathfrak{Fol}'_{\Gamma,X}(n)$ (*optimal Følner set*) then it is connected as subgraph of the right Cayley graph of Γ with respect to X (see [2] for details). In particular for $f \in F'$ we have that $1_\Gamma \in F' := F'f^{-1}$ and $F' \subset B_{|F'|}(\Gamma)$ and $F' \in \mathfrak{Fol}'_{\Gamma,X}(n)$. Finally since:

$$\mathfrak{Fol}_{\Gamma,X}(|X|n) \subset \mathfrak{Fol}'_{\Gamma,X}(n) \subset \mathfrak{Fol}_{\Gamma,X}(n)$$

we have that $F \in \mathfrak{Fol}_{\Gamma,X}(n)$ and $|F| \leq F_{\Gamma,X}(|X|n)$. \square

Definition 1. Let y_1, y_2, \dots, y_s be pairwise commuting elements of Γ , not necessarily distinct. Set:

$$C_n(y_1, y_2, \dots, y_s) := \{y_1^{i_1} y_2^{i_2} \dots y_s^{i_s} : i_1, i_2, \dots, i_s \in \{0, 1, \dots, n-1\}\}.$$

Lemma 3.

$$\frac{|C_n(y_1, y_2, \dots, y_s) \setminus y_j C_n(y_1, y_2, \dots, y_s)|}{|C_n(y_1, y_2, \dots, y_s)|} \leq n^{-1}, \quad \forall j \in \{1, 2, \dots, d\}.$$

Proof. Since all elements y_1, y_2, \dots, y_s commute we prove, without loss of generality, the statement for $j = 1$.

At first, we observe that $C_n(y_1, y_2, \dots, y_s) = C_n(y_1)C_n(y_2, y_3, \dots, y_s)$ and

$$C_n(y_1) \setminus y_1 C_n(y_1) = \begin{cases} \emptyset & \text{if } y_1 \text{ has order less than or equal to } n \\ \{1_\Gamma\} & \text{otherwise.} \end{cases}$$

Writing C_n instead of $C_n(y_1, y_2, \dots, y_s)$ we have that

$$C_n \setminus y_1 C_n \subset C_n(y_2, y_3, \dots, y_s),$$

because $C_n \setminus y_1 C_n \subset [C_n(y_1) \setminus y_1 C_n(y_1)]C_n(y_2, y_3, \dots, y_s)$.

Now we show that C_n contains n disjoint translations of $C_n \setminus y_1 C_n$, precisely:

$$(2) \quad C_n \supset \bigsqcup_{k=0}^{n-1} y_1^k [C_n \setminus y_1 C_n].$$

At first

$$y_1^k [C_n \setminus y_1 C_n] \subset y_1^k C_n(y_2, y_3, \dots, y_s) \subset C_n, \quad \forall k \in \{0, 1, \dots, n-1\};$$

in particular if $g \in y_1^k [C_n \setminus y_1 C_n]$ there exist $\hat{i}_2, \dots, \hat{i}_s \in \{0, 1, \dots, n-1\}$ such that $g = y_1^k y_2^{\hat{i}_2} \dots y_s^{\hat{i}_s}$.

If $k \neq 0$ then $g \notin C_n \setminus y_1 C_n$, this implies:

$$y_1^k [C_n \setminus y_1 C_n] \cap [C_n \setminus y_1 C_n] = \emptyset, \quad \forall k \in \{1, \dots, n-1\}.$$

Thus $\{y_1^k [C_n \setminus y_1 C_n]\}_{k=0, \dots, n-1}$ are disjoint sets and (2) is proved and therefore we deduce $\frac{|C_n \setminus y_1 C_n|}{|C_n|} \leq n^{-1}$.

□

For a finite subset $Y \subset \Gamma$ we may have different finite enumerations of Y , for example we consider $W, W' \subset \mathbb{F}_X$, $W = \{w_1, \dots, w_t\}$ and $W' = \{w'_1, \dots, w'_{t'}\}$ such that $\pi_{\Gamma, X}(W) = \pi_{\Gamma, X}(W') = Y$. In general, $C_n(\pi_{\Gamma, X}(w_1), \dots, \pi_{\Gamma, X}(w_t)) \neq C_n(\pi_{\Gamma, X}(w'_1), \dots, \pi_{\Gamma, X}(w'_{t'}))$ in Γ but these subsets are both n^{-1} -invariant by left multiplication by every element $y \in Y$, by virtue of Lemma 3. By abuse of notation we simply write $C_n(Y)$ instead of $C_n(\pi(w_1), \dots, \pi(w_t))$ when the choice of the finite preimage W of Y is irrelevant.

Moreover, when the generating subset $X \subset \Gamma$ is clear from the context, we shall simply write F_Γ (resp. $\mathfrak{F}\phi_\Gamma$) instead of $F_{\Gamma, X}$ (resp. $\mathfrak{F}\phi_{\Gamma, X}$).

1. KHARLAMPOVICH GROUPS

Theorem 1.1. *Let $\Gamma = \langle L_1 \cup L_2 \rangle$ be a finitely generated group, L_1 and L_2 two finite disjoint subsets and respectively H_1 and H_2 the subgroups that they generate. Suppose that H_2 is amenable, H_1^Γ is Abelian and $\Gamma = H_1^\Gamma \rtimes H_2$, then:*

$$AC_n(L_1^A) \in \mathfrak{Fol}_\Gamma(n), \quad \forall A \in \mathfrak{Fol}_{H_2}(n).$$

where $L_1^A = \{a^{-1}xa : a \in A, x \in L_1\}$.

Proof. Set $B := C_n(L_1^A)$, and observe that $|AB| = |A||B|$ since $A \subset H_2$ and $B \subset H_1^\Gamma$ and $H_2 \cap H_1^\Gamma = \{1_\Gamma\}$.

For $x \in L_2$ we have:

$$\frac{|AB \setminus xAB|}{|AB|} \leq \frac{|A \setminus xA||B|}{|A||B|} \leq n^{-1},$$

For $x \in L_1$, using Lemma 3, we have:

$$\begin{aligned} \frac{|AB \setminus xAB|}{|AB|} &= \frac{|\{ab : a \in A, b \in B : ab \notin xAB\}|}{|A||B|} = \\ &= \frac{|\{ab : a \in A, b \in B : b \notin a^{-1}xAB\}|}{|A||B|} \leq \frac{|\{ab : a \in A, b \in B : b \notin a^{-1}xaB\}|}{|A||B|} \leq \\ &\leq \frac{|\bigcup_{a \in A} a(B \setminus a^{-1}xaB)|}{|A||B|} \leq n^{-1} \text{ (since } a^{-1}xa \in L_1^A \text{ and } B = C_n(L_1^A)). \end{aligned} \quad \square$$

Consider the description of a Kharlampovich group $G(M)$ given in [13], with M a Minsky machine with unsolvable halting problem and p a fixed prime, using the same notation of [13], we have:

- $H_j := \langle L_j \rangle$, $j = 0, 1, 2$, and $H := \langle L_1 \cup L_2 \rangle$,
- H_j is abelian, $j = 0, 1, 2$;
- H_0, H_1 are of exponent p ;
- H_1^H is abelian of exponent p ;
- $H = H_1^H \rtimes H_2$;
- $H_0^{G(M)}$ is abelian of exponent p ;
- $G(M) = H_0^{G(M)} \rtimes H$.

Then by Theorem 1.1 we have:

$$C_n(L_2) \in \mathfrak{Fol}_{H_2}(n),$$

$$C_n(L_2)C_n(L_1^{C_n(L_2)}) \in \mathfrak{Fol}_H(n),$$

since H_1^H is of exponent p , then for $n \geq p$ we have $C_n = C_p$ in H_1^H and the same holds in $H_0^{G(M)}$, thus:

$$C_n(L_2)C_p(L_1^{C_n(L_2)})C_p(L_0^{C_n(L_2)C_p(L_1^{C_n(L_2)})}) \in \mathfrak{Fol}_{G(M)}(n).$$

The groups $G(M)$ have computable Følner sets: we have an algorithm with input n and output a finite subset of the free group projecting onto an n -Følner set in $G(M)$. The Theorem A of Introduction is proved. Moreover, we have a bound from above for the cardinality of the smallest Følner sets for $G(M)$.

Corollary 1.2. *The class of finitely presented groups with computable Følner sets is larger than the class of finitely presented amenable groups with solvable Word Problem.*

Corollary 1.3.

$$F_{G(M)}(n) \leq n^{|L_2|} p^{|L_1|n^{|L_2|}} p^{|L_0|n^{|L_2|}p^{|L_1|n^{|L_2|}}}.$$

2. ABELIAN EXTENSION

We consider now the general Abelian extensions: a priori the procedure doesn't ensure computability of the Følner sets in every case.

Theorem 2.1. *Let Γ be finitely generated by X . Suppose $N \triangleleft \Gamma$ is an Abelian normal subgroup and denote by $\rho : \Gamma \rightarrow \Gamma/N$ the canonical projection. Then*

$$AC_{2n|A|^2}(A^{-1}XA \cap N) \in \mathfrak{Fol}_{\Gamma, X}(n),$$

for each finite $A \subset \Gamma$ such that $|A| = |\rho(A)|$ and $\rho(A) \in \mathfrak{Fol}_{\Gamma/N, \rho(X)}(2n)$.

Proof. Consider the finite set $S := A^{-1}XA \cap N$ and, for each $x \in X$, the finite set $S_x := A^{-1}xA \cap N$. We clearly have $|S| \leq |A|^2|X|$ and $|S_x| \leq |A|^2$.

Set $B := C_{2n|A|^2}(S) \subset N$. Then by Lemma 3 we have $\frac{|B \setminus sB|}{|B|} \leq (2n|A|^2)^{-1}$ for all $s \in S$;

thus for any $s \in S_x$, for any $x \in X$

$$(3) \quad \frac{|B \setminus sB|}{|B|} \leq (2n|S_x|)^{-1}.$$

Consider the set $F := AB \subset \Gamma$ and notice that $|F| = |A||B|$ because the intersection $A \cap B$ has at most one element since $\rho|_A$ is injective and ρ sends B to the identity of Γ/N . So for $g \in F$ we write $g = ab$, $a \in A$, $b \in B$ in a unique way (again because $\rho|_A$ is injective and $\rho(g) = \rho(a)$)

and we write $A' := \rho(A) \subset \Gamma/N$, recall that this is $2n$ -Følner in Γ/N .

For each $x \in X$, the set $F \setminus xF$ is the disjoint union of the subsets:

$$E_1^x = \{g \in F \setminus xF : \rho(g) \notin \rho(x)A'\}$$

$$E_2^x = \{g \in F \setminus xF : \rho(g) \in \rho(x)A'\}.$$

If $g = ab \in E_1^x$, since $\rho(g) = \rho(a) \notin \rho(x)A'$ we have $\rho(a) \in A' \setminus \rho(x)A'$. But ρ is injective on A then:

$$(4) \quad \frac{|E_1^x|}{|F|} = \frac{|A' \setminus \rho(x)A'| |B|}{|A| |B|} \leq (2n)^{-1}.$$

If $g = ab \in E_2^x$ then $\rho(a) \in \rho(x)A' = \rho(xA)$. Hence there exist $a' \in A, s \in N$ such that $as = xa'$. It follow that $s = a^{-1}xa'$ and $s \in S_x$. Now $g = xa's^{-1}b$, and since $g \notin xF = xAB$ we necessarily have $b \notin sB$. Thus we have

$$\frac{|E_2^x|}{|F|} \leq \frac{|\{xa's^{-1}b, a' \in A, s \in S_x, b \in B \setminus sB\}|}{|A| |B|} \leq \sum_{s \in S_x} \frac{|B \setminus sB|}{|B|}$$

And by (3):

$$(5) \quad \frac{|E_2^x|}{|F|} \leq (2n)^{-1}.$$

Combining (4) and (5) we deduce that $\frac{|F \setminus xF|}{|F|} = \frac{|E_1^x|}{|F|} + \frac{|E_2^x|}{|F|} \leq n^{-1}$, for any $x \in X$. \square

Corollary 2.2. *A finitely presented group which is the extension of an amenable group with solvable Word Problem by an Abelian group has computable Følner sets.*

Proof. Consider the case of Γ/N amenable with solvable Word Problem and with the set $\rho(X)$ as generators. If $\pi_{\Gamma/N} : \mathbb{F}_X \rightarrow \Gamma/N$ is the canonical epimorphism, for every n we can compute $\mathcal{A} \in \mathbb{F}_X$ such that $\pi_{\Gamma/N}(\mathcal{A}) \in \mathfrak{Fol}_{\Gamma/N, \rho(X)}(2n)$, but also with $|\mathcal{A}| = |\pi_{\Gamma/N}(\mathcal{A})|$, by the solvability of the Word Problem.

But then $A := \pi_{\Gamma, X}(\mathcal{A})$ is such that $\rho(A) = \pi_{\Gamma/N}(\mathcal{A}) \in \mathfrak{Fol}_{\Gamma/N, \rho(X)}(2n)$ and $|A| = |\rho(A)|$, because:

$$|\rho(A)| \leq |A| \leq |\mathcal{A}| = |\pi_{\Gamma/N}(\mathcal{A})|.$$

Moreover, given an element $\omega \in \mathcal{A}^{-1}XA$ we can compute if $\pi_{\Gamma/N}(\omega) = 1_{\Gamma/N}$ or not, and then we can compute the preimage of $A^{-1}XA \cap N$ in \mathbb{F}_X and finally we can compute a preimage of the n -Følner sets for Γ . \square

This implies again that Kharlampovich groups have computable Følner sets, because they are Abelian extensions of finitely presented metabelian, and therefore residually finite with solvable WP, groups.

Notice that the Abelian group N may be not finitely generated.

Corollary 2.3. *If Γ is finitely generated by X and $N \triangleleft \Gamma$ is an Abelian normal subgroup, denoting with $\rho : \Gamma \rightarrow \Gamma/N$ the projection:*

$$F_\Gamma(n) \leq F_{\Gamma/N}(2n)(2nF_{\Gamma/N}(2n)^2)^{|X|F_{\Gamma/N}(2n)^2}.$$

Proof. We consider $\rho(A) \in \mathfrak{Fol}_{\Gamma/N}(2n)$ such that $|\rho(A)| = |A| = F_{\Gamma/N}(2n)$, recall that $S = A^{-1}XA \cap N$ and then $|S| \leq |X||A|^2$. \square

3. SPLITTING EXTENSIONS

The situation is clearer if the extension splits. In this case we can also consider extensions by amenable groups.

Theorem 3.1. *Let N and H be groups respectively generated by the finite sets Z and Y , let $\phi : H \rightarrow \text{Aut}(N)$ be a homomorphism. Let $c := \max\{|\phi_y(z)|_Z : z \in Z, y \in Y\}$. Then if $A \in \mathfrak{Fol}_{H,Y}(n)$ and $B \in \mathfrak{Fol}_{N,Z}(nc^{|A|_Y})$ we have*

$$AB \in \mathfrak{Fol}_{N \rtimes_\phi H, Z \cup Y}(n),$$

(recall that $|A|_Y = \max\{|a|_Y : a \in A\}$).

Proof. We first observe that $|AB| = |A||B|$ because $A \subset H$ and $B \subset N$.

For $y \in Y (\subset H)$ we have:

$$\frac{|AB \setminus yAB|}{|AB|} \leq \frac{|A \setminus yA||B|}{|A||B|} \leq n^{-1}.$$

For $z \in Z (\subset N)$ we have:

$zab = aa^{-1}zab = a\phi_a(z)b$, so that $\{ab \in AB : zab \notin AB\} \subset \{ab \in AB : \phi_a(z)b \notin B\}$. We deduce

$$\frac{|AB \setminus zAB|}{|AB|} \leq \frac{|\bigcup_{a \in A} a[B \setminus \phi_a(z)B]|}{|A||B|} \leq \frac{\sum_{a \in A} |B \setminus \phi_a(z)B|}{|A||B|}.$$

Since $|\phi_a(z)|_Z \leq c^{|a|_Y} \leq c^{|A|_Y}$ then, using Lemma 1:

$$\frac{\sum_{a \in A} |B \setminus \phi_a(z)B|}{|A||B|} \leq \frac{|\phi_a(z)|_Z}{c^{|A|_Y}n} \leq n^{-1},$$

because $B \in \mathfrak{Fol}_N(nc^{|A|_Y})$. \square

Corollary 3.2. *The semidirect product of two finitely generated groups with computable Følner sets has computable Følner sets.*

Proof. We can compute \mathcal{A} , the preimage of a n -Følner set A for H , we compute m , the maximal length of words in \mathcal{A} in the free group. We compute \mathcal{B} , the preimage of $B \in \mathfrak{Fol}_N(nc^m)$. Since $|A|_Y \leq m$ we have $B \in \mathfrak{Fol}_N(nc^{|A|_Y})$ and then by Theorem 3.1 we have that \mathcal{AB} is a preimages of an n -Følner set for the semidirect product. \square

Corollary 3.3. *In the same hypotheses of the above theorem:*

$$F_{N \rtimes_{\phi} H}(n) \leq F_H(n|Y|)F_N(nc^{F_H(n|Y|)}).$$

Proof. By Lemma 2 we have $A \in \mathfrak{Fol}_H(n)$ with $|A|_Y \leq |A| \leq F_H(|Y|n)$ then we choose the optimal $B \in \mathfrak{Fol}_N(nc^{F_H(n|Y|)})$. Clearly $B \in \mathfrak{Fol}_N(nc^{|A|_Y})$. \square

4. GENERAL EXTENSIONS

Theorem 4.1. *Let Γ be generated by the finite set X and N be a normal subgroup of Γ generated by the finite set Y . Let $\rho : \Gamma \rightarrow K := \Gamma/N$ be the projection to the quotient. For any finite subset $A \subset \Gamma$ such that $A' := \rho(A) \in \mathfrak{Fol}_{K, \rho(X)}(2n)$, with $|A| = |A'|$ and $|A|_X \leq |A'|_{\rho(X)}$, and any $B \in \mathfrak{Fol}_{N, Y}(2n|A'|^2 \Delta_N^{\Gamma}(2|A'|_{\rho(X)} + 1))$ we have*

$$AB \in \mathfrak{Fol}_{\Gamma, X}(n).$$

Proof. Setting $F := AB$ it is easy to see that $|F| = |A'||B|$ because ρ is injective on A .

For each $x \in X$, the set $F \setminus xF$ is the disjoint union of the sets E_1^x and E_2^x , defined by:

$$E_1^x = \{g \in F \setminus xF : \rho(g) \notin \rho(x)A'\}$$

$$E_2^x = \{g \in F \setminus xF : \rho(g) \in \rho(x)A'\}.$$

We can write $g = ab$, with $a \in A$ and $b \in B$, in a unique way.

If $g \in E_1^x$, since $\rho(g) = \rho(a) \notin \rho(x)A'$ we have $\rho(a) \in A' \setminus \rho(x)A'$. Moreover, since ρ is injective on A :

$$\frac{|E_1^x|}{|F|} = \frac{|A' \setminus \rho(x)A'| |B|}{|A'| |B|} \leq (2n)^{-1}.$$

If $g \in E_2^x$ then $\rho(g) = \rho(a) \in \rho(x)A'$ so that there exists $a' \in A$ satisfying $\rho(a) = \rho(x)\rho(a')$.

The images by ρ of a and xa' are the same so we can find $s \in N$ such that $as = xa'$.

Setting $S_x := A^{-1}xA \cap N$ we see that $s \in S_x$ and $|S_x| \leq |A|^2$. Then $g = xa's^{-1}b$, and since $g \notin xAB$ we deduce that $b \notin sB$. It follows that:

$$\frac{|E_2^x|}{|F|} \leq \frac{|\{xa's^{-1}b, a' \in A, s \in S_x, b \in B \setminus sB\}|}{|A'| |B|} \leq \sum_{s \in S_x} \frac{|B \setminus sB|}{|B|}.$$

We have a bound for $|S_x|$; we need a bound for the length of the elements in S_x . For every $s \in S_x$ we have:

$|s|_Y \leq \Delta_N^\Gamma(|s|_X)$. On the other hand, $|s|_X = |a^{-1}xa'|_X \leq 2|A|_X + 1 \leq 2|A'|_{\rho(X)} + 1$.

From Lemma 1 we then deduce:

$$\frac{|B \setminus sB|}{|B|} \leq (2n|A'|^2)^{-1} \leq \frac{1}{2n|S_x|}.$$

Finally $\frac{|F \setminus xF|}{|F|} = \frac{|E_1^x|}{|F|} + \frac{|E_2^x|}{|F|} \leq n^{-1}$, showing that F is an n -Følner set. \square

Corollary 4.2. *Let N, Γ, K finitely generated groups such that:*

$$1 \rightarrow N \rightarrow \Gamma \rightarrow K \rightarrow 1.$$

If N has computable Følner sets, Δ_N^Γ is subrecursive, K is amenable with solvable Word Problem, then Γ has computable Følner sets.

Proof. N and K have computable Følner sets. For each k we can construct $\mathcal{A} \subset \mathbb{F}_X$ such that $\pi_K(\mathcal{A}) \in \mathfrak{Fol}_K(k)$. We denote $A' := \pi_K(\mathcal{A})$. If we consider $A := \pi_{\Gamma, X}(\mathcal{A})$, it is clear that $\rho(A) = A' \in \mathfrak{Fol}_K(k)$. If K has solvable Word Problem we can detect \mathcal{A} such that π_K is injective on \mathcal{A} and $|\omega| = |\pi_K(\omega)|_{\rho(X)}$ for every $\omega \in \mathcal{A}$. So we can compute a preimage for a set A respecting the hypotheses of the Theorem 4.1. For the set B we just need the computability (of a bound) of the number $2n|A'|^2 \Delta_N^\Gamma(2|A'|_{\rho(X)} + 1)$, so if Δ_N^Γ is subrecursive we have the thesis. \square

Finally, from Theorem 4.1 and again using Lemma 2:

Corollary 4.3. *Let N, Γ, K finitely generated groups such that:*

$$1 \rightarrow N \rightarrow \Gamma \rightarrow K \rightarrow 1.$$

Then

$$F_\Gamma(n) \leq F_K(|X|n) F_N(2n F_K(|X|n)^2 \Delta_N^\Gamma(2F_K(|X|n) + 1)).$$

Thus if N and K have subrecursive Følner function and if Δ_N^Γ is subrecursive then Γ has subrecursive Følner function as well.

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(M. Cavaleri) INSTITUTE OF MATHEMATICS OF THE ROMANIAN ACADEMY, 21 CALEA GRIVITEI STREET,
010702 BUCHAREST, ROMANIA

E-mail address: matte.cavaleri@gmail.com